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Nonlocal theory of the high-strain-rate processes in structured media

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Abstract

A new nonlocal hydrodynamic approach to describe structured media is developed. According to this approach the nonlocal and spin properties of a medium are closely correlated. The concrete kind and scale of the medium structure resulting from the strain process are defined by the initial and boundary conditions in a nonunique way due to the branching of solutions to the nonlinear problem. As a consequence, in the same medium localization of the strain process can be realized either in the form of shear banding or rotational motion. As a test task the well-known Rayleigh problem on nonsteady motion of a plate in viscous media is solved to show that the degree of nonlocality is proportional to acceleration of the plate. The solution obtained is then used to explain experimental results on shock-induced shear bands and vortex structures in metals. © 1999 Elsevier Science Ltd. All rights reserved.

Nomenclature

λ, L	typical lengths of an internal structure and hydrodynamical flow
g	hydrodynamical variable
ε, τ	nonlocality and memory scale parameters
G	hydrodynamical gradients
P	dissipative fluxes
$\mathscr{L}$	relaxation transport kernels
k	transport coefficients
S, σ	parameters of the model transport kernels
φ	deviation from the Navier–Stokes value of a hydrodynamical variable
D, **L**, **J**	linearized operators
B	boundary of a system
$\mathscr{G}$	Green function

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Φ	nonlinear functional
Δ	deviation from the Navier–Stokes value of an integral characteristic
ν	kinematic viscosity
u	shear velocity
U, $\dot{U}$	velocity and acceleration of the plate
y	coordinate normal to the plate
$\mathcal{M}$	memory function
α, γ	concrete model parameters for the Rayleigh problem

1. Introduction

A number of processes of high-velocity straining in solids are known often to have quite definitive features of hydrodynamical flow. First of all it concerns an interaction of shape charge and target, high-velocity penetration of long rods and so on. The limits of applicability of hydrodynamical model conception in the theory of high-velocity collisions of solids have been analyzed comprehensively (see, for example Hohler and Stilp, 1990; Zlatin and Kozhushko, 1982). In particular, analysis performed in the paper (Zlatin and Kozhushko, 1982) and comparison of its results with well-known experimental data permits one to conclude that the concept of ideal incompressible liquid may be used for the high-velocity interaction of solids to be described in a quite narrow range of penetrator velocities. For example, use of the Bernoulli equation gives an acceptable precision in calculations of crater formation processes within the impactor velocity range, where the strength of material has no influence but the compressibility still influences the results of collision simulations. This requires extension of the hydrodynamical model for adequate description of these processes.

On the other hand, the use of the hydrodynamical description of processes in solids even under lower impact velocities, where as a rule the elastic–plastic description is commonly used, can also be justified due to the fact that during the shock wave passage the material is in the so-called unstable state.

In both cases the medium flow shares the common properties and require the general approach to describe:

(i) the kind and scale of elementary carrier of deformation for the shock wave propagation problem and the kind and scale of structure elements for liquid flow must be defined by the boundary and initial conditions of the nonlinear problem in a self-consistent manner.
(ii) The kind of kinematical mechanism for dynamic straining (translational or rotational) as well as for hydrodynamical flow must also be defined by the imposed conditions and must automatically change when the strain rate changes.

The above considerations constitute a base to extend the hydrodynamical description for the high-velocity processes in dynamically loaded solids.

In this paper an attempt is made to show that in order to extend the hydrodynamical description it is necessary to transit from ideal liquid approach, and even from classical local approach, to non-local hydrodynamics of structured liquids. In Section 2 on the basis of the latest experimental data on high-velocity interaction of solids and shock wave propagation processes, the main principles and requirements for the sought theory are formulated. In Section 3 the theoretical basis for the self-consistent nonlocal description of medium with internal structure is given in details. Finally, in Section 4 the model problem on a non-steady motion of infinite plate in viscous structured medium (the so-called Rayleigh problem) is considered for revealing the most specific features of medium flow. It is shown, in particular, that relative accelerations of structure elements define both the non-local effects and the effects related to the previous history of straining. The obtained results confirm that the new theory presents an adequate description of the non-equilibrium processes in solids under dynamical loading and describes the experimental results without introducing any empirical parameters.

2. Experimental evidence of nonlocal approach

The hydrodynamical theory of high-velocity interaction of solids has to take into account mechanical features of target and impactor: viscosity, compressibility, strength. Alekseevskii (1966) and Tate (1967) have independently included into the Bernoulli equation dynamic strength of target and impactor materials:

$$Y+\tfrac{1}{2}\rho_p(v-u)^2 = \tfrac{1}{2}\rho_\tau u^2+R. \tag{2.1}$$

Here Y is the dynamic strength of impactor material and R is the same for the material of target, v and u are the particle velocities in the material of penetrator and target, respectively. By using Y and R as fit parameters the theory of Alekseevskii (1966) and Tate (1967) enable the experimental penetration curve for an arbitrary projectile–target–material combination to be described. However, if the same target–material is used in combination with different projectile materials, the R value will not be constant and equal in each and every case. The same may be said about Y, if only the target material is changed. This means that the theory based on the modified Bernoulli equation and which deals with ideal incompressible fluid with artificial empirical parameters, cannot be considered as adequate describing even the quasistatic stage of penetration process.

Some alternative description of high-velocity interaction of solids seems to be a using of Ginsburg–Landau equation:

$$\tfrac{1}{2}\rho_\tau u^2+\rho_T v_T\dot{\varepsilon}+\tfrac{1}{2}\rho_T c_{0T}^2 = \tfrac{1}{2}\rho_p(v-u)^2+\rho v\dot{\varepsilon}_p+\tfrac{1}{2}\rho_p c_{0p}^2 \tag{2.2}$$

where $c_T = \sqrt{c_l \cdot c_t}$, c_l and c_t are the velocities of transverse and longitudinal elastic waves. As distinct from eqn (2.1), eqn (2.2) does not involve the fit parameters. However, as has been shown in paper (Balankin, 1991), that equation has a strictly limited range of application defined by continuity of viscosity value for penetrator and target materials. In reality, as it is seen for D-16 alloy (Fig. 1), in the process of high-velocity straining the viscosity can change more than two orders.

One of the main reasons for theory and experiment discrepancy seems to be a contradiction between phenomenological models affecting the smooth change of strength with the shock velocity increasing and fundamental laws of thermodynamics of the irreversible processes. In particular, phenomenological models cannot take into account the tendency of open systems, which are far from equilibrium state, to self-organization by means of formation of the so-called dissipative

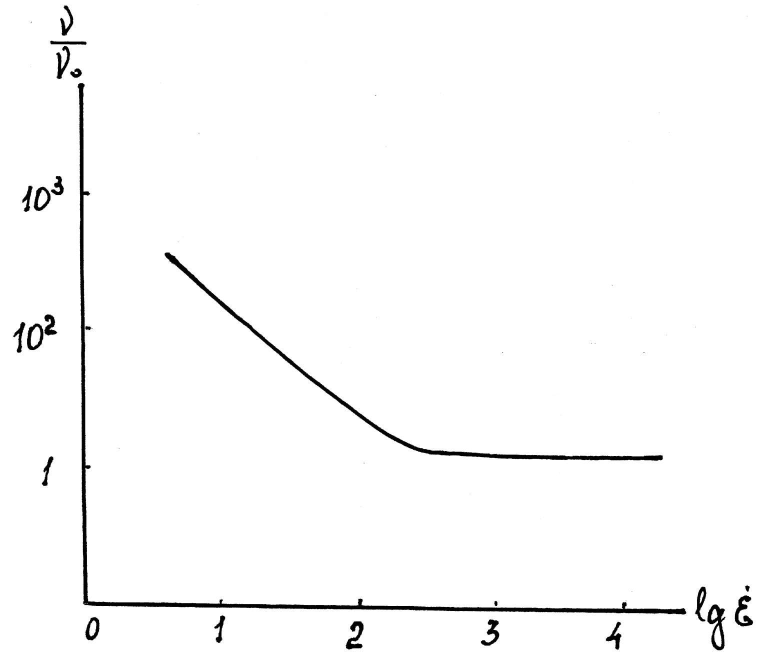

Fig. 1. Dependence of viscosity on the strain rate in D-16 aluminum alloy (after Balankin).

structures. During the high-velocity interaction of penetrator–target couple, which is certainly a typical open system, change of the kinematical mechanism of straining can occur under some strain-rate conditions. Typical example testifying the sudden change of resistivity of material accompanied by changing the kinematical mechanism is presented in Fig. 2. One can see, that under velocity of 900–1100 m/s the penetration length in Pb and Cu increases in a step-like manner. This phenomenon cannot be described in the framework of the hydrodynamical model of ideal incompressible liquid even if the fit parameters Y and R are introduced into the model.

The transition from laminar to turbulent flow has also been found during the interaction of plane shape charges and annealed Cr–Ni–Mo steel (Savenkov et al., 1990). Analogous phenomenon was found during the spall-strength tests of the same steel (Barakhtin et al., 1991). Under impactor velocity of 350 m/s the spall strength of material suddenly increases. Microstructure investigations of specimens after shock tests reveal the change of kinematical mechanism of deformation. Instead of shear banding which can be classified as a translational mechanism of dynamic straining at the mesoscopical scale level (0.1–10 μm), numerous rotational cells of the same scale level have been found.

The applicability of hydrodynamical approach for the high-velocity processes in solids even under lower impact velocities can be justified due to the fact that during the wave passage the material comes into the so-called unstable state. Evidence for the structure instability of solids during the propagation of elastic–plastic waves can be seen, for example, from the micrograph, presented in Fig. 3. It demonstrates the chain of rotational cells (vortexes) in copper target loaded

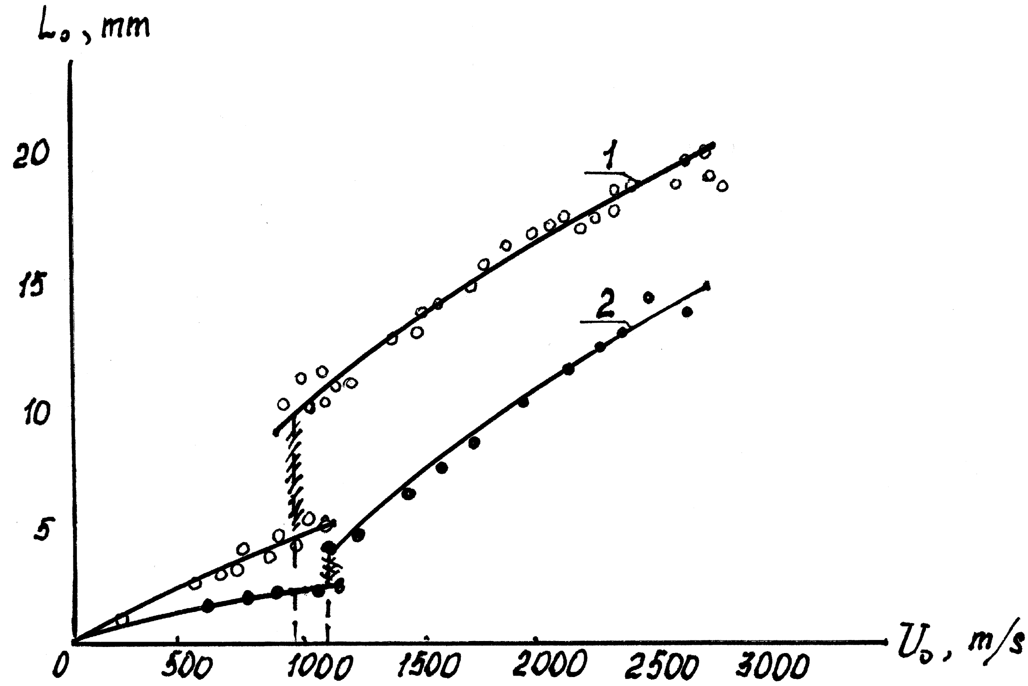

Fig. 2. Dependence of penetration depth on the impact velocity after collision of copper disk with the target from: 1. lead, 2. copper (after Balankin).

under uniaxial strain conditions with the impact velocity of 160 m/s (Mescheryakov et al., 1992, 1994). Element analysis of inner structure of rotational cells shows that they consist of the same material as the matrix. This means that these cells represent the "frozen" vortex formations of material itself (not inclusions). One of the remarkable features of these rotational chains is that, being elongated in the wave propagation direction, they cross the grain boundaries without change of their direction (see micrograph in Fig. 4). This certainly means that during the plastic front passage material is really in an unstable state, and grain boundaries disappear as strong obstacles for the motion of microflows. Narrow region between adjacent microflows of different velocities is then transformed either into shear band or rotational (vortex) cells depending on the degree of nonlocality of dynamic deformation process. An analogous situation is known to appear on the boundaries of two layers in liquid—the so-called "cat-eyes" formations in the Helmholtz instability phenomenon. It was also pointed in (Mescheryakov et al., 1992, 1994) that the cross-section of rotational cells along the chain changes non-monotonously—their dimension reaches two maximums with gradual decreasing to the edges and middle of the chain (see Fig. 3). Sometimes, in the middle of a chain instead of rotations there is a short shear band. Comparison with positions of maximums for time-resolved profile of particle velocity accelerations curve obtained by differentiating the particle velocity dispersion (see Fig. 5) permits the conclusion that the size of rotation correlates with the particle acceleration. At the same time, shear banding occurs when acceleration changes its sign and particle velocity dispersion is maximum. On the basis of these experimental data it has been concluded that the difference in acceleration of microflows is responsible for the rotational mechanism of localization, while the difference in particle velocities is responsible for shear band.

Although the aforedescribed liquid-like behavior of medium was found in a shock compressed solid (copper) this phenomenon cannot be described in the framework of traditional elastic–plastic theory. On the other hand, it also cannot be described using classical hydrodynamics of ideal liquid or Navier–Stokes equations. These equations are local, valid for structureless media and do not involve the change of flow regime in a wide range of imposed conditions.

Thus, we come to the necessity to extend the hydrodynamical approach and to take into account the aforedescribed phenomena. This supposes that the sought theory, being hydrodynamical, would be capable to provide both rearrangements of structure scales and change of kinematical mechanisms of flow. In particular, it must provide as well a transition from laminar to turbulent flow in the case of high-velocity penetration processes as transition from shear banding to rotational motion of medium in case of shock wave propagation processes.

The sought theory must be self-consistent, i.e. must take into account an influence of overall deformed volume, including the boundaries, on material flow in a local region. The parameters of that theory (scale of structure elements, viscosity, degree of micropolarization for the momentum media) must integrally depend both on the boundary conditions and whole flow process while accounting for the balance relations for density, momentum and energy.

The enumerated requirements lead to the necessity to develop an extended nonlocal hydrodynamical description of structured media.

3. Self-consistent nonlocal hydrodynamical approach to describe nonequilibrium transport processes in structured media

3.1. The scale parameters for structured media on highly nonequilibrium conditions

The classical continuum model of media allows satisfactory description of flows of real media on sufficiently large space-time scales far from the critical points and phase transitions. According to this model the state of a system is characterized by mean-field qualities which are assumed to be governed by the differential balance equations. In such a treatment one considers the conservation laws for a medium element which has a linear size l negligible in comparison with the typical flow length L but much exceeding the scale λ of the medium internal structure:

$$\lambda \ll l \ll L.$$

In general the additional scale is considered to have the magnitude of a typical radius of nonlocal correlations due to the collective interaction of medium structure elements. Its size may vary from the mean free path of a particle in gas to the linear scale of the medium inner structure and in concentrated dispersed media may enclose rather many particles. Generally, the macroscopical scale L is assumed to be the inhomogeneity scale for a macroscopical variable g:

$$L = \frac{g}{|\mathrm{grad}\, g|}.$$

In case of a slight inhomogeneity L can be taken as being equal to the typical length of a system.

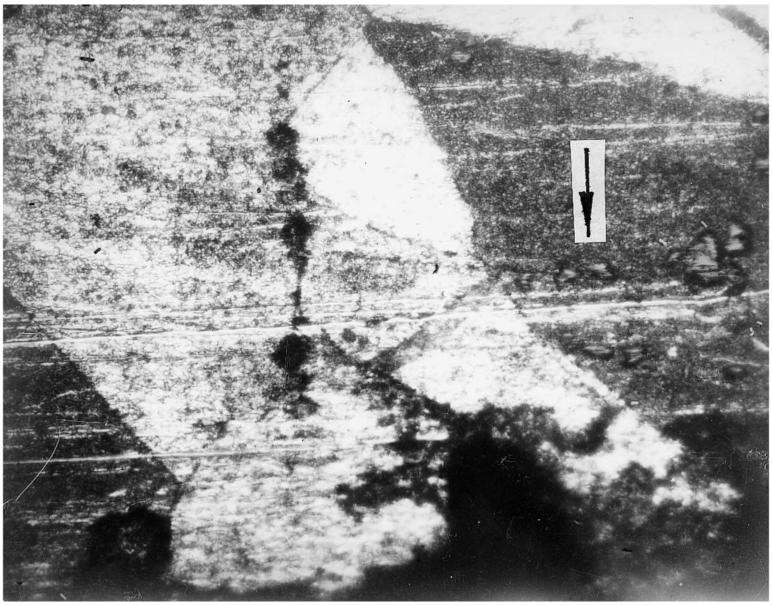

Fig. 3. Micrograph of rotation chain in copper. Arrow shows the shock wave propagation direction (magnification × 320).

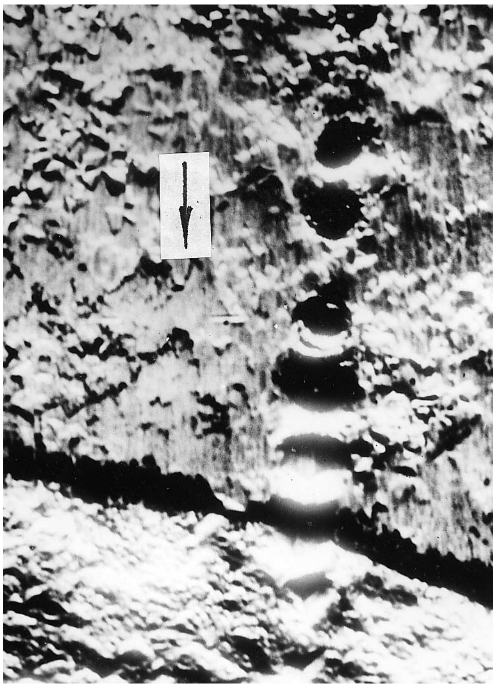

Fig. 4. Micrograph of rotation chain which crosses a grain boundary. Arrow shows the shock wave propagation direction (magnification × 1000).

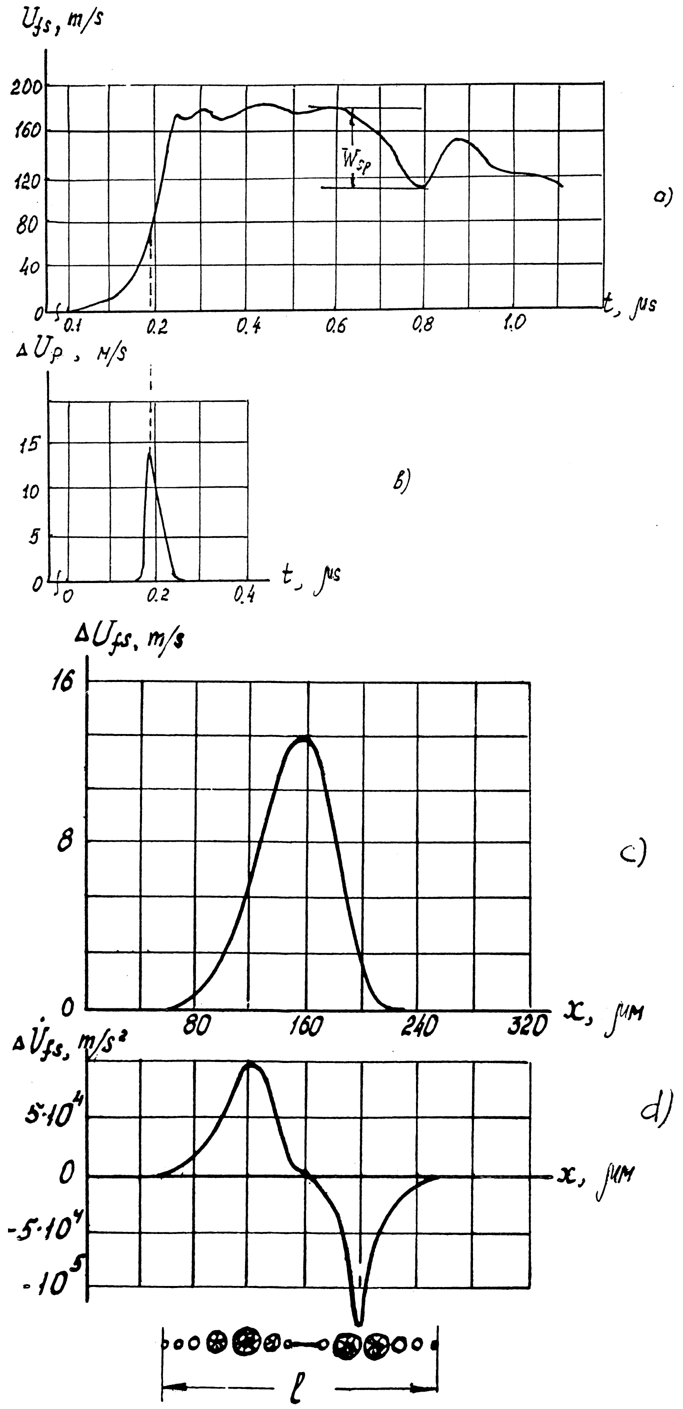

Fig. 5. (a) Free surface velocity profile in copper target loaded by copper impactor under velocity of 160 m/s. (b) Time history of the particle velocity distribution width. (c) Space profile of the particle velocity distribution width. (d) Space distribution of time derivative of the particle velocity distribution width and positions of its maxima relative rotation chain.

So we must introduce a parameter whose value is closely connected with a choice of descriptional level for real problems:

$$\varepsilon = \lambda/L$$

then the limit $\varepsilon \to 0$ corresponds to the usual continuum mechanics when the real medium structure effects can be neglected. The basis of this theory is the assumption that the thermodynamical state of the macroscopical system is near to the local equilibrium one. In the framework of the linear irreversible transport thermodynamics the hypothesis about the linear dependencies between the dissipative fluxes and macroscopical gradients is the simplest assumption permitting completion of the macroscopical balance equations. In this manner the main equations of the theoretical hydrodynamics—Navier–Stokes equations—have been obtained. They are not valid to govern flows with large hydrodynamical gradients and high velocity processes, when the internal structure effects become essential and the usual transport coefficients become inadequate. The more inhomogeneity of a medium (less the scale L) the more the internal structure effects (more the scale λ), the more the parameter ε increases and the nonequilibrium of a system increases. In this situation macroscopical balance equations are not entirely localized. They imply nonlocal in space and time constitutive relationships between macroscopical gradients G and dissipative fluxes P:

$$P(r,t) = \int_V \mathrm{d}r' \int_0^t \mathrm{d}t' \mathscr{L}(r,r',t,t';\varepsilon,\tau) G(r',t') + P_0(r,t), \tag{3.1}$$

where weight factors $\mathscr{L}$ represent the relaxation transport kernels. In general they are defined by unknown functions of the hydrodynamical densities and depend on space and time scale parameters ε, τ.

If a deviation from the local equilibrium is small, the scale parameters tend to zero: ε, $\tau \to 0$. The relaxation kernels reduce to the transport coefficients, and the relationships between gradients and fluxes become local and linear

$$P(r,t) \sim k(r,t)G(r,t) \equiv P^0(r,t), k(r,t) \equiv \lim_{\substack{\varepsilon\tau\to 0\\ \tau\to 0}} \int_V \mathrm{d}r' \int_0^t \mathrm{d}t' \mathscr{L}(r,r',t,t';\varepsilon,\tau). \tag{3.2}$$

In this sense the scale parameters can be taken as being the nonlocality and memory parameters.

3.2. Nonlocal relationships between the dissipative fluxes and the hydrodynamical gradients

By using the expression (3.2) we can rewrite eqn (3.1) as follows

$$P(r,t) = \int_V \mathrm{d}r' \int_0^t \mathrm{d}t' \tilde{\mathscr{L}}(r,r',t,t';\varepsilon,\tau) P^0(r',t') + P_0(r,t). \tag{3.3}$$

Further we shall omit the symbol "~" above $\mathscr{L}$.

These relationships had been derived from the first principles in the nonequilibrium statistical mechanics (Richardson, 1960; Piccirelli, 1968; Zubarev and Tischenko, 1972). In the derivation of the nonlocal relations it was shown, that any calculations of the relaxation transport kernels as of the time correlation functions required solving simultaneously the set of generalized hydrodynamics engaged with the microscopical dynamic equations. Herewith, if the transport kernels or the correlation functions are considered as given time-space functions the microscopical set is discarded, the whole set splits and the hydrodynamical equations become self-contained. Taking different explicit expressions for the kernels we can get governing relationships for media with different properties and construct convenient models for any practical problems. However, all attempts to introduce the nonlocal hydrodynamics as a theoretical background into the usage of the calculating hydrodynamics and applied mechanics did not become common until the present because of a large gap in their descriptional levels and approaches.

It must be pointed out a circumstance of great importance. Both the nonequilibrium statistical operator method and all other methods proceeding from the Liouville equation are valid only for isolated systems. However, it is the nonequilibrium stationary states which are maintained by the imposed fixed boundary conditions, that are of prime interest to practice.

3.3. *Self-consistent nonlocal models and their essential features*

A new trend of the nonlocal hydrodynamics developed by one of the authors of this paper (Khantuleva, 1982, 1984, 1992; Khantuleva and Vavilov, 1994) is a construction of self-consistent models of the relaxation transport kernels. Herewith, it is supposed that the boundary effects connecting the interaction of an open system with its surroundings, can be involved as an additional element of modeling for the relaxation transport kernels. The nonlocal models must correspond to general invariability and asymptotic principles and depend on a minimal quantity of parameters.

Based on these principles a δ-type class of the relaxation transport kernels depending on parameters was defined. The model parameters are connected with the influence of the internal structure on hydrodynamics as a whole. The physical sense of these parameters was examined by means of a test problems for which it is possible to compare the results with those obtained in the kinetic theory and with the experimental data.

To account for the spatial nonlocality along the x-direction a simple model of the δ-type kernel is proposed:

$$\mathscr{L}(x', x; \varepsilon) = \frac{S(x;\varepsilon)}{\varepsilon}\,\omega\left(\frac{|x' - x - \sigma(x;\varepsilon)|}{\varepsilon}\right). \tag{3.4}$$

The model (3.4) corresponds to the requirement of a uniform limiting transition to the continuum mechanics when $\varepsilon \to 0$ up to the boundaries. It must be pointed out that the limiting transition is fixed: $\varepsilon \to 0$, $\sigma \to 0$. The expression (3.4) is easily generalized for the three-dimensional case taking into account the fact that the nonlocal scales along the different directions ε_i can differ one from the other.

The model (3.4) includes the parameters:

ε is a nonlocality parameter or the relative space correlation scales for hydrodynamical densities;
$\omega(|\xi|;\varepsilon) \geqslant 0$ is a δ-type function of $|\xi|$ depending on ε as on a parameter defining a rate of space relaxation of hydrodynamical correlations;

$\sigma(x;\varepsilon)$ is a shift parameter making the model anisotropic and connected with polarization effects incited by large hydrodynamical gradients including the boundary effects;
$S(x;\varepsilon)$ is a normalizing factor characterizing the structure average effects, such as an effective transport coefficient $S \to 1$ when $\varepsilon \to 0$.

It has been shown that a medium of structure elements with finite linear size under inhomogeneous field of stresses and $\varepsilon \sim 1$ becomes anisotropic, there exists spin motion due to an asymmetrical stress tensor (Aero, 1981).

This aspect of the approach has close connections with the theory of multipolar fluids (Green and Rivlin, 1964; Bellout et al., 1992). But the proposed theory involves the spin properties of structure elements implicitly due to the nonlocal reduced description which makes it differ from the above-mentioned one. Generally, in the construction of nonlocal models with $\varepsilon \sim 1$ we have to take into account the polarization effects in a medium with highly nonequilibrium conditions.

According to the self-consistent approach to the construction of the model relaxation transport kernels the model parameters are related to any integral properties of a system either through integral relationships including such characteristics as a flow rate, sum momentum and energy, or by imposing additional boundary conditions. These additional relationships for the model parameters complete a set of the nonlocal equations and make the formulation of the boundary problem self-consistent. The self-consistence of the proposed approach is its special feature which is followed by very important consequences.

The essential property of the approach is a preservation in the generalized hydrodynamical equations of the integral information about a system in the description of the local hydrodynamical fields. This circumstance in a radical way changes the conception of the boundary-value problems in the nonlocal theory. Unlike the classical continuum models the self-consistent nonlocal models are uniformly valid up to the boundaries. Thus, the solutions provided by these equations can satisfy the real boundary conditions considered to be the continuity conditions for the hydrodynamical fields. It means that on solid boundaries we can use the non-slip conditions even for highly nonequilibrium flows when the classical continuum models lead to discontinuities on boundaries or near them.

In as much as the additional functional boundary conditions making the self-consistent model closed can be imposed rather arbitrarily, this approach allows prediction of the conditions for formation of space structures with *a priori* predicted properties. This is a very important advantage of the self-consistent nonlocal models, which may be used in a wide range of technological applications. Due to the branching process arising in a nonlinear system on nonequilibrium conditions the state of a system can change discontinuously as the conditions of external interaction change smoothly. It means that the proposed approach gives a new possibility to describe the structural transitions.

The main advantage of the proposed approach consists in an evident flexibility, efficiency and universality which allows applications in fluid mechanics, space engineering, chemical and microelectronic technologies, synergetics and ecology.

3.4. Slightly nonlocal approximation

Consider asymptotic expansions of the integrals (3.3) with the model kernels (3.4) when $\varepsilon \to 0$ in the one-dimensional case for a semi-axis $x \in [0, \infty]$. Taking the Taylor series of a function P in the integral (3.3) in a vicinity of a point $x' = x$, we obtain a differential constitutive relationship instead of the integral one:

$$P(x;\varepsilon) = S(x;\varepsilon)\sum_{n=0}^{\infty}\frac{l_n(x;\varepsilon)}{n!}\frac{\partial^n P^0}{\partial x^n}(x);$$

$$l_n(x;\varepsilon) \equiv \int_{-[(x+\varepsilon)/\varepsilon]}^{\infty} \mathrm{d}\xi\omega(|\xi|)(\varepsilon\xi+\sigma)^n$$

$$= \sum_{k=0}^{n}\frac{n!(a_{n-k}+r_{n-k}(x;(\varepsilon))}{(n-k)!k!}\sigma^k(x;\varepsilon)\varepsilon^{n-k};$$

$$a_p = \begin{cases} M_\omega^P(-\infty) = \text{const} & p = 2m, \\ 0, & p = 2m+1, \end{cases}$$

$$M_\omega^P(z) \equiv \int_z^{\infty} \mathrm{d}\xi\omega(|\xi|)\xi^P$$

$$r_p(x;\varepsilon) = \begin{cases} -M_\omega^P\left(\dfrac{x+\sigma}{\varepsilon}\right) = \text{const} & p = 2m, \\ M_\omega^P\left(\dfrac{x+\sigma}{\varepsilon}\right)^{\varepsilon}, & p = 2m+1 \end{cases} \tag{3.5}$$

In order to go over to a differential operator of the finite order N we need to evaluate the N-th term of the series supposing $\varepsilon \ll 1$. Then we have to require an order relation $l_k \sim \varepsilon^k$ satisfied uniformly all over the flow region up to the boundaries. Here some connections between the model parameters follow:

$$l_0 = a_0 = 1, \quad \sigma \sim \varepsilon, \quad r_k \sim \varepsilon^k, \quad x \in [0, \infty].$$

Then taking these connections into account we get a uniform asymptotic approximation of the order $N = N(\varepsilon)$, $\varepsilon \ll 1$:

$$P^N(x;\varepsilon) = S(x;\varepsilon)\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{a_{n-k}}{(n-k)!k!}\sigma^k(x;\varepsilon)\varepsilon^{n-k}\frac{\partial^n P^0}{\partial x^n}(x). \tag{3.6}$$

The parameters S, σ can be expanded too:

$$S(x;\varepsilon) = 1+\sum_i s_i(x)\varepsilon^i; \quad \sigma(x;\varepsilon) = \sum_i \sigma_i(x)\varepsilon^i.$$

The first approximation is

$$P^1(x;\varepsilon) = (1+\varepsilon s_1(x))P^0+\varepsilon\sigma_1(x)\frac{\partial P^0}{\partial x}. \tag{3.7}$$

It may be called the slightly nonlocal model. The presence of a small parameter at the highest derivative is an indication of possible singularities such as the boundary layers and shock waves of a thickness $O(\varepsilon)$.

So, for gases the expansion (3.6) with the cross-effects between velocity and temperature gradients being included can be considered as a generalization of the Chapman–Enskog procedure for the case of boundary-value problems. Then the corresponding integral expression is regarded as being its convolution.

3.5. Reducing of the self-consistent model formulation for the boundary-value problems to a special type operator set

Write a set of nonstationary hydrodynamical equations for compressible viscous heat-conducting medium with the nonlocal expressions for the viscous stress tensor and heat flux vector and with the model dependencies for the relaxation transport kernels (3.4) in a symbolic operator form

$$\mathbf{D}_g = \mathbf{J}(S, \sigma; \varepsilon)g; \tag{3.8}$$

where $\mathbf{D}_g$ is a nonlinear differential operator of the first-order corresponding to the convective parts of the Navier–Stokes equations; $\mathbf{J}(S, \sigma, \varepsilon)$ is a nonlinear integro-differential operator corresponding to the dissipative irreversible parts of the generalized equations and depending on the model parameters S, σ and on the nonlocality parameter ε through the asymmetrical δ-type kernels; g are hydrodynamical variables (density, mass velocity, temperature).

For the Navier–Stokes equations which results from the set (3.8) in the limit $\varepsilon \to 0$, the following boundary conditions must be usually satisfied

—non-slip condition on solid boundaries:

$$r = r_B, \quad g = g_B, \tag{3.9}$$

—free stream condition:

$$(r)_n \to \infty, \quad g \to g_\infty, \tag{3.10}$$

—initial conditions:

$$t = 0, \quad g = g_c. \tag{3.11}$$

Bearing in mind that solutions to the set of nonlocal eqns (3.8) must not be discontinuous on boundaries we have to assume the conditions (3.9)–(3.11) holding in the case $\varepsilon \sim 1$. If we use the Navier–Stokes approximation in this case we should obtain incorrect values of the local friction and the heat transfer. In order to define these values correctly we need to take into account the nonlocal effects at $\varepsilon \sim 1$ through which the internal structure effects are included in the hydrodynamical equations.

So, at $\varepsilon \sim 1$ the boundary-value problem at hydrodynamical level is formulated as a set of eqns (3.8) with the model constitutive relationships (3.3)–(3.4) and the boundary conditions (3.9)–(3.11). Herewith the main difficulty to use such a model consisted in the absence of any mathematical ground to formulate and to solve such boundary-value problems. However, these problems appear to have analogs in the theory of the resonance problems in mechanics, to which they can be reduced by usual methods.

Let us seek corrections φ to the hydrodynamical variables in the Navier–Stokes approximation g_0 bearing in mind the latter is correct only when $\varepsilon \ll 1$: $g = g_0 + \varphi$. Then we can linearize the left-hand parts of the set (3.8) and rewrite it in a form

$$\mathbf{D}_0 \varphi = \mathbf{L}_0(g_0, \varphi) + \mathbf{J}(S, \sigma; \varepsilon)(g_0, \varphi) \tag{3.12}$$

where $\mathbf{D}_0$ is a linearized convective operator and $\mathbf{J}$ is nonlinear integro-differential operator.

Unlike the second-order Navier–Stokes equations, the nonlocal ones in the form (3.12) are first-order differential equations. This circumstance is followed by the fact that not all of the boundary conditions (3.9)–(3.11) can be satisfied.

Supposing the Navier–Stokes values satisfy the conditions (3.9)–(3.11), the corrections must satisfy the uniform boundary conditions

$$\varphi_B = 0. \tag{3.13}$$

In order to find φ from a set (3.12) formally we can use only one part of these boundary conditions

$$\varphi_{B1} = 0, \quad B1 \bigcup B2 = B. \tag{3.14}$$

If we know the Green function for the uniform operator equation $D_0\varphi = 0$ on condition $\varphi_{B1} = 0$, we can represent the eqn (3.12) in the integral form

$$\varphi(r, t) = \int_V \mathrm{d}r' \int_0^t \mathrm{d}t' \mathscr{G}(r, r', t, t')[\mathbf{L}_0(g_0; \varphi) + \mathbf{J}(S, \sigma; \varepsilon)(g_0, \varphi)](r', t'). \tag{3.15}$$

Herewith we are left with the boundary conditions $\varphi_{B2} = 0$ not being satisfied. We must remember that according to the self-consistent nonlocal model the parameters S, σ are assumed not arbitrary but any functionals of the sought solution itself. Hence we can require the remainder part of the boundary conditions (3.13) to be satisfied on account of the model parameters S, σ. Substituting the formal solution (3.15) into the condition $\varphi_{B2} = 0$ we get a functional relationship including the model parameters S, σ

$$\int_V \mathrm{d}r' \int_0^t \mathrm{d}t' \mathscr{G}(r, r', t, t')[\mathbf{L}_0(g_0; \varphi) + \mathbf{J}(S, \sigma; \varepsilon)(g_0, \varphi)](r', t') = 0. \tag{3.16}$$

It is very important that in general the boundary conditions by the eqn (3.16) determine not all tensor components of the parameters S, σ. The deficit relations can be obtained by imposing any additional functional conditions connected with the integral properties of a system such as a flow rate, sum momentum and energy and others. Herewith the given integral values must explicitly include deviations from their Navier–Stokes values Δ

$$\Phi[g_0, \varphi](\Delta, S, \sigma; \varepsilon) = 0. \tag{3.17}$$

It must be pointed out that eqns (3.16)–(3.17) determine either constant values of the parameters S, σ or reduce their coordinate dimension if the integrals are taken with respect to only some coordinates.

Now we have got a closed formulation of the boundary value problem: an operator eqn (3.15) and the functional relationships (3.16)–(3.17) determining the model parameters.

This formulation can be represented in a general form

$$u = F(u, \xi); \quad \Phi_j(u, \xi) = 0, \quad j = 1, \ldots, n \tag{3.18}$$

with respect to unknown element $u \in E$ in Banach space and involved n model parameters $\xi \in \mathbf{R}^n$. F is a nonlinear operator and Φ_j are nonlinear functionals.

In order to analyze the solvability problem for the (3.18)-type operator set there have been developed two methods: the geometrical one being based on Galerkin approximation scheme and the comparison method based on a simple iteration process. Both methods give universal mathematical grounds to examine a wide class of various physical situations. By using developed methods, the solvability conditions of the (3.18)-type operator set have been formulated and the algorithms to construct approximate solutions valid for boundary problems are well established. If these conditions are satisfied the iteration procedures converge to a precise solution (Vavilov, 1990, 1992; Vavilov and Yuhnevich, 1993).

These methods also allow successive analysis of the branching process for the nonlinear problems while complicated practical problems cannot sometimes be analyzed by the classical methods.

4. Nonstationary shear flow of structured media

4.1. Memory effects in shear flow

Nonstationary hydrodynamical equations in the Navier–Stokes approximation for the structured medium flows have some essential deficiencies. These equations being of the parabolic type cannot describe flows at small typical times in principle. The last circumstance makes it difficult to state the Cauchy problem because in this situation it becomes necessary to establish dummy initial conditions which differ from the real ones.

Let us consider the Rayleigh problem which is believed to be typical for an analysis of nonstationary effects. An infinite plane at time $t = 0$ is instantaneously driven in motion at a constant velocity U_0. Then a motion of a viscous structureless medium is governed by the parabolic equation for the shear velocity $u(y, t)$:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{4.1}$$

where $\nu = \mu_0/\rho_0 = \text{const}$ is kinematic shear viscosity. The initial and boundary conditions follows:

$$u(y, t = 0) = 0, \quad u(y = 0, t = 0) = U_0, \quad u(y \to \infty, t \geqslant 0) \to 0. \tag{4.2}$$

A well-known solution to this problem has a form

$$u(y, t) = U_0 \left(1 - \text{erf} \frac{y}{\sqrt{4\nu t}}\right) \equiv u^0(y, t); \tag{4.3}$$

$$P(y, t) = -\nu \frac{\partial u}{\partial y} = \frac{U_0 \nu}{\sqrt{\pi \nu t}} e^{-\frac{y^2}{4\nu t}} \equiv P^0(y, t). \tag{4.4}$$

This solution has a δ-type singularity $P^0(y, t) \to 2\delta(y)\nu U_0$, $t \to 0$ which results from the parabolic type of eqn (4.1). If $\partial u/\partial t\ (y, t = 0) \neq \partial u^0/\partial t(y, t = 0)$, there exists an initial layer, where eqn (4.1) is not correct. In this case we have to take into account the memory effects.

Generally, in order to do this it is necessary to introduce a so-called memory function $1/\tau\mathscr{M}(t/\tau)$ with a parameter τ having sense of a relaxation time. The memory function is defined by the internal structure effects in a medium which are essential at the macroscopic level at a given typical velocity of a process. When $\tau \to 0$, the memory function becomes of the type: $1/\tau\mathscr{M}(t/\tau) \to \delta(t)$. Then instead of the parabolic eqn (4.1) we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}(y,0)\mathscr{M}(t/\tau) + \nu \int_0^t \frac{\mathrm{d}t'}{\tau}\mathscr{M}\left(\frac{t-t'}{\tau}\right)\frac{\partial^2 u}{\partial y^2}(y,t'). \tag{4.5}$$

When $t \to 0$ and $(\partial u/\partial t)(y, 0) = (\partial u^0/\partial t)(y, 0)$, eqn (4.5) tends to eqn (4.1) and then $u \to u^0$. Now we shall reduce the problem (4.5) under conditions (4.2) to the operator form (3.18) and shall seek a solution as follows. By integrating eqn (4.5) with respect to time and supposing on the right hand side we get a first approximation

$$u_1 = \frac{\partial u}{\partial t}(y,0)\int_0^t \mathscr{M}(t'/\tau)\,\mathrm{d}t' + \nu \int_0^t \mathrm{d}t' \int_0^{t'} \frac{\mathrm{d}t''}{\tau}\mathscr{M}\left(\frac{t'-t''}{\tau}\right)\frac{\partial^2 u}{\partial y^2}(y,t''), \tag{4.6}$$

which at the plate ($y = 0$) gives:

$$u_1(0,t) = \frac{\partial u}{\partial t}(0,0)\int_0^t \mathscr{M}(t'/\tau)\,\mathrm{d}t'. \tag{4.7}$$

If the velocity of the plate is given as a time-dependent function $U(t)$, eqn (4.7) can be rewritten as follows:

$$U(t) = \dot{U}(0)\int_0^t \mathscr{M}(t'/\tau)\,\mathrm{d}t'. \tag{4.8}$$

Equation (4.8) means that the plate is driven in motion not instantaneously but with finite acceleration. When the value $\dot{U}(0)$ is finite we can use the expression (4.8) to determine the memory function through the known history of the acceleration by differentiating eqn (4.8) with respect to time

$$\dot{U}(t) = \dot{U}(0)\mathscr{M}(t/\tau), \quad \mathscr{M}(t) = \frac{\dot{U}(t)}{\dot{U}(0)}. \tag{4.9}$$

This determination of the memory function is available in the case when the integral structure effects are essential only on time scales small compared with the structure relaxation time during the acceleration stage of the process. After this stage $U(t) \to U_0$. Then we can determine the relaxation time as follows:

$$\tau = U_0/\dot{U}(0) \to 0, \quad \dot{U}(0) \to \infty; \tag{4.10}$$

$$\frac{1}{\tau}\mathscr{M}(t) = \frac{\dot{U}(t)}{U_0} \to \delta(t). \tag{4.11}$$

It means that neglecting the values $O(\tau)$ we get the classical case of a plate instantaneously driven in motion.

Rewrite the expression (4.6) in terms of the memory function (4.9)–(4.11)

$$u_1(y,t) = \frac{\partial u}{\partial t}(y,0)\frac{U(t)}{\dot{U}(0)} + v\int_0^t \mathrm{d}t' \int_0^{t'} \frac{\mathrm{d}t''}{U_0}\dot{U}(t'-t'')\frac{\partial^2 u}{\partial y^2}(y,t'')$$

$$= \theta(y)U(t) + \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \frac{\dot{U}(t'-t'')}{U_0}\frac{\partial u^0}{\partial t''}. \tag{4.12}$$

By expanding into the Taylor series the function $\partial u^0/\partial t''$ in the vicinity of the point $t'' = t'$ and retaining only the first term we get

$$u_1(y,t) \approx \theta(y)U(t) + \int_0^t \frac{U(t')}{U_0}\frac{\partial u^0}{\partial y'}(y,t')\,\mathrm{d}t'. \tag{4.13}$$

Equation (4.13) gives an approximate solution to the nonstationary generalization of the Rayleigh problem without the memory effects. The solution (4.13) satisfies the non-slip boundary condition at $y = 0 \forall t$ and when $U(t) \to U_0$, tends to the classical solution u^0. It is worth noticing that the solution (4.13) can be obtained otherwise using the Green function for the parabolic eqn (4.1) with the source

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} + \int_0^t \mathrm{d}t'\left(\frac{\dot{U}(t)}{U_0} - 1\right)\frac{\partial u^0}{\partial t'} + \theta(y)\dot{U}(t);$$

$$u_1 = u^0 + \int_0^t \mathrm{d}t' \int_0^\infty \mathrm{d}\xi \int_0^{t'} \mathrm{d}t'' \frac{\left(\frac{\dot{U}(t)}{U_0} - 1\right)\frac{\partial u^0}{\mathrm{d}t''} + \theta(\xi)\dot{U}(t'')}{2\sqrt{\pi v(t'-t'')}}$$

$$\cdot \exp\left\{-\frac{(y-\xi)^2}{4v(t'-t'')}\right\} \underset{v \ll 1}{\approx} \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\theta(y)\dot{U}(t'') + \frac{\dot{U}(t'')}{U_0}\frac{\partial u^0}{\partial t''}\right]$$

$$= \theta(y)U(t) + \int_0^t \mathrm{d}t' \frac{U(t)}{U_0}\frac{\partial u^0}{\partial t'}. \tag{4.14}$$

With a simple exp-type form for the memory function instead of eqn (4.5) we get

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}(y,0)\,e^{-t/\tau} + v\int_0^t \frac{\mathrm{d}t'}{\tau} e^{-\frac{t-t'}{\tau}} \frac{\partial^2 u}{\partial y^2}(y,t'). \tag{4.15}$$

Taking the derivative with respect to time in eqn (4.15) the telegraph equation is derived

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} - \tau\frac{\partial^2 u}{\partial t^2}. \tag{4.16}$$

This equation is hyperbolic and describes a wave damping process on a given initial condition

$\partial u/\partial t(y, 0) \neq 0$. In the framework of the developed theory instead of the solution (4.6) we can get an approximate solution

$$u_1 = \frac{\partial u}{\partial t}(y,0)\tau(1-e^{-t/\tau}) + v\int_0^t \mathrm{d}t' \int_0^{t'} \frac{\mathrm{d}t''}{\tau} e^{-\frac{t'-t''}{\tau}} \frac{\partial^2 u}{\partial y^2}(y, t''). \tag{4.17}$$

Then at the plate $y = 0$ we have

$$\frac{\partial u_1}{\partial t}(0, t) = \dot{U}(0)\, e^{-t/\tau}. \tag{4.18}$$

Herewith, in case when a plate is driven in motion with acceleration $\dot{U}(t)$, the solution (4.17) remembers only the initial value $\dot{U}(0)$ which according to eqn (4.18) decays exponentially. If the function $\partial^2 u/\partial y^2(y, t'')$ in eqn (4.17) is expended into the Taylor series in a vicinity of the point $t'' = t'$ we can get a solution up to the value $O(\tau)$:

$$u_1 \approx \tau\left(\frac{\partial u}{\partial t}(y,0) - \frac{\partial u^0}{\partial t}(y,0)\right) + u_0 + \tau\frac{\partial u^0}{\partial t} + O(e^{-t/\tau}) \underset{\tau\to 0}{\to} u^0. \tag{4.19}$$

So, approximate solutions to eqn (4.14) appear to describe the internal structure effects as memory which take place only at finite accelerations. As the structure effects can be displayed also through nonlocality, it is interesting to analyze their correlation with accelerations.

4.2. *Nonlocal effects in shear flow*

Now consider the nonlocal extension of the problem for a structured medium flow with the nonlocality parameter $\varepsilon \sim 1$ (see Section 3.1):

$$\frac{\partial u}{\partial t} = v(1+\alpha)\frac{\partial}{\partial y}\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon} e^{-\frac{\pi}{\varepsilon^2}(y'-y-\gamma)^2} \frac{\partial u}{\partial y'}(y', t). \tag{4.20}$$

The shear component of the viscous stress tension takes a form

$$P(y,t) = -v(1+\alpha)\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon} e^{-\frac{\pi}{\varepsilon^2}(y'-y-\gamma)^2} \frac{\partial u}{\partial y'}(y', t) = (1+\alpha)\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon} e^{-\frac{\pi}{\varepsilon^2}(y'-y-\gamma)^2} P^0(y', t). \tag{4.21}$$

The model parameters α, γ are to be defined. Here instead of the model parameters S, σ we introduce new ones: $S = 1+\alpha$, $\sigma = \gamma$, where α, γ are considered to be independent of y. As $\varepsilon \to 0$ and also $\alpha, \gamma \to 0$, eqn (4.20) leads to eqn (4.1) and $P \to P^0$.

If we want to use the classical solution as the O-order approximation for the nonlocal problem we must remember that without the memory effects we cannot describe the initial stage of the plate's motion and satisfy the real initial conditions. That is why we can formulate the following problem: at time $t = 0$ the plate is instantaneously driven in motion with the velocity $U(t)$

$$U_\varepsilon(t) \underset{\substack{t\to\infty \\ \varepsilon\to 0}}{\longrightarrow} U_0 = \text{const}, \quad U_\varepsilon(0) \neq U_0, \quad U_\varepsilon(0) \neq 0.$$

In such a way we can trace the terminal stage of acceleration.

Now rewrite eqn (4.20) to separate the nonlocal effects as follows:

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u(y,t)}{\partial y^2} + N_\varepsilon(y,t;\alpha,\gamma), \tag{4.22}$$

$$N_\varepsilon(y,t;\alpha,\gamma) \equiv v\frac{\partial}{\partial y}\left[(1+\alpha)\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon} e^{-\frac{\pi}{\varepsilon^2}(y'-y-\gamma)^2}\frac{\partial u}{\partial y'} - \frac{\partial u}{\partial y}\right]. \tag{4.23}$$

In the slightly nonlocal approximation (see Section 3.4) the expression (4.23) can be written in a form

$$N_\varepsilon(y,t;\alpha,\gamma) = v\left[\alpha\frac{\partial^2 u}{\partial y^2} + \gamma\frac{\partial^3 u}{\partial y^3}\right] \underset{\alpha,\gamma\to 0}{\to} 0. \tag{4.24}$$

We shall seek a solution to eqn (4.20) under conditions (4.2), where $U_\varepsilon(t)$ is taken instead of U_0, in a form $u = u^0 + \varphi$, $\varphi(0,t) = \varphi(\infty,t) = 0 \forall t \geqslant 0$. In order to reduce the problem to the operator form (3.18) we have to take the Green function for the parabolic eqn (4.2) with a source N_ε

$$\varphi(y,t) = v\int_0^t\int_0^\infty \frac{N_\varepsilon(\xi,\theta;\varphi(\xi,\theta);\alpha,\gamma)}{2\sqrt{\pi v(t-\theta)}} e^{-\frac{(y-v)^2}{4v(t-\theta)}}\,\mathrm{d}\xi\,\mathrm{d}\theta. \tag{4.25}$$

The Green function being of the δ-type we can get an approximate formal solution when $v \ll 1$:

$$\varphi(y,t) = v\int_0^t N_\varepsilon(y,\theta;\varphi(y,\theta);\alpha(\theta),\gamma(\theta))\,\mathrm{d}\theta. \tag{4.26}$$

In order to define the model parameters α, γ we use first the boundary condition $\varphi(0,t) = 0$, ($\varphi(\infty,t) = 0$ being satisfied)

$$v\int_0^t N_\varepsilon(0,\zeta;\varphi(0,\zeta);\alpha(\zeta),\gamma(\zeta))\,\mathrm{d}\zeta = 0. \tag{4.27}$$

It must be pointed out that in this approximation the model parameters α, γ can be determined only as the time-dependent functions $\alpha(t)$, $\gamma(t)$. That is why, in order to define one of them in a correct way we must differentiate eqn (4.27) with respect to t:

$$N_\varepsilon(0,t;\varphi(0,t);\alpha(t),\gamma(t)) = 0. \tag{4.28}$$

Equation (4.28) is equivalent to eqn (4.27) up to the constant. It will be clear further that the constant is equal to $U_0 = \int_0^\infty \dot{U}(t)\,\mathrm{d}t$ in a first approximation.

In addition, we need one more condition to close the boundary value problem and to define the model parameters α, γ. Suppose we know from the experiments or from any other considerations the time dependence of the viscous friction on a plate

$$P(0,t) = -\int_0^\infty \frac{\partial u}{\partial t}(y,t)\,\mathrm{d}y - P_W(t) \neq P^0(0,t). \tag{4.29}$$

Then from (4.28) and (4.29) we have two branching equations for the sought parameters α, γ,

$$\Phi_\varepsilon^1(\varphi;\alpha,\gamma) \equiv v(1+\alpha)\int_0^\infty \left[\frac{\partial}{\partial y} e^{-\frac{\pi}{\varepsilon^2}(y'-y-\gamma)^2}\right]_{y=0} \frac{\partial(u^0+\varphi)}{\partial y'} - \dot{U}(t) = 0; \tag{4.30}$$

$$\Phi_0^2(\rho;\alpha,\gamma) \equiv v(1+\alpha)\int_0^\infty \frac{\partial y'}{\varepsilon} e^{-\frac{\pi}{\varepsilon^2}(y'-\gamma)^2} \frac{\partial(u^0+\varphi)}{\partial y'} - P_W(t) = 0. \tag{4.31}$$

Now the boundary problem for eqn (4.22) and given dependences at $y = 0$ is reduced to the operator eqn (4.26) and functional relationships (4.30)–(4.31) which entirely correspond to the operator formulation (3.18).

In the first approximation the problem is written as follows:

$$\varphi^1(y,t) = v\int_0^t \mathrm{d}t(1+\alpha)\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon}\frac{\partial}{\partial y} e^{-\frac{\pi}{\varepsilon^2}(y'-\gamma)^2} \frac{\partial u^0}{\partial y'}(y',t) - y^0(y,t); \tag{4.32}$$

$$\dot{U}(t) = v(1+\alpha)\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon}\left[\frac{\partial}{\partial y} e^{-\frac{\pi}{\varepsilon^2}(y'-\gamma)^2}\right]_{y=0} \frac{\partial u^0}{\partial y'}; \tag{4.33}$$

$$P_W(t) = v(1+\alpha)\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon} e^{-\frac{\pi}{\varepsilon^2}(y'-\gamma)^2} \frac{\partial u^0}{\partial y'}. \tag{4.34}$$

From eqns (4.33) and (4.34), we obtain the nonlinear equation for the sought function $\gamma(t)$, the so-called branching equation

$$\frac{\displaystyle\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon}\left[\frac{\partial}{\partial y} e^{-\frac{\pi}{\varepsilon^2}(y'-\gamma)^2}\right]_{y=0} \frac{\partial u^0}{\partial y'}}{\displaystyle\int_0^\infty \frac{\mathrm{d}y'}{\varepsilon} e^{-\frac{\pi}{\varepsilon^2}(y'-\gamma)^2} \frac{\partial u^0}{\partial y'}} = \frac{\dot{U}(t)}{P_W(t)}. \tag{4.35}$$

In general eqn (4.35) can have more than one solution. In the asymptotic case when using the δ-type properties of the integral kernels, we obtain the unique values of the parameters α, γ and, corresponding to them, the unique solution to the boundary problem in the first approximation:

$$\gamma(t) = \frac{\dot{U}(t)}{U_0}\frac{2\sqrt{\pi v t^3}}{\Delta(t)}; \quad \Delta(t) \equiv \frac{P_W(t)}{P^0(0,t)}; 1+\alpha(t) = \Delta(t)\exp\left(\frac{\dot{U}(t)}{U_0^2}\frac{\pi t^2}{\Delta^2(t)}\right). \tag{4.36}$$

As $U(t) \xrightarrow[t\to\infty]{} U_0$, or $\dot{U}(t) \to 0$, and $\Delta(t) \to 1$, the parameters $\alpha(t)$, $\gamma(t) \to 0$ and $U^1 \to U^0$, $P^1 \to P^0$.

Further we can linearize the relations (4.36) near the solution using the slightly nonlocal approximation (see Section 3.4). In this case the formulation (4.32)–(4.34) becomes more simple

$$\varphi(y,t) = v\int_0^t \mathrm{d}t\left\{\alpha(t)\frac{\partial^2 u^0}{\partial y^2} + \gamma(t)\frac{\partial^3 u^0}{\partial y^3}\right\}; \tag{4.37}$$

$$\dot{U}(t) = v\gamma(t)\frac{\partial^3 u^0}{\partial y^3}(0,t); \tag{4.38}$$

$$P_W(t) = (1+\alpha(t))P^0(0,t). \tag{4.39}$$

Substituting the known values of the velocity gradients into eqn (4.37)–(4.39) we have got a new explicit solution to the boundary problem (4.27), (4.30)–(4.31) as $\varepsilon \to 0$:

$$\varphi(y,t) = \int_0^t \mathrm{d}t\left\{(\Delta(t)-1)\frac{\partial u^0}{\partial t} + \frac{\dot{U}(t)}{U_0}\sqrt{\pi v^3 t^3}\,\frac{\partial^3 u^0}{\partial y^3}\right\}. \tag{4.40}$$

The asymptotical values of the model parameters are unique:

$$\gamma(t) = \frac{\dot{U}}{U_0}\sqrt{vt^3}; \quad \alpha(t) = \frac{P_W^0(t) - P^0(0,t)}{P^0(0,t)}.$$

Herewith, the shift parameter of the kernel $\gamma(t)$ is a measure of the thickness of a near-boundary layer which is related to the structure effects and can be much thinner than the classical Prandtl layer.

5. Discussions of the obtained results and conclusions

First conclusions of the analysis performed in Sections 4.1–4.2 are enumerated:

(1) in scope of the proposed approach the integral structure effects manifest themselves as the memory and nonlocal effects. The special feature of the nonstationary processes is the fact that both effects emerge only at finite accelerations of a plate $U(t) \neq 0$;
(2) the nonlocal effects at movements with accelerations cause the medium polarization along a direction normal to a plate (a direction of the greatest gradients). It is found that the shift parameter $\gamma \neq 0$ in the momentum relaxation transport kernel is directly proportional to the plate acceleration: $\gamma \sim \dot{U}$. A medium becomes anisotropic at the cost of an asymmetry of the viscous stress tensor;
(3) the medium anisotropy may follow from the emergence of turning moments of the medium structure elements. According to the results of the paper (Aero, 1981) the turning moments appear in the medium composed of finite size elements under the influence of nonuniform stresses. Herewith an acceleration $\dot{U}(t) \neq 0$ is responsible for the rotations of structure elements generated by the nonlocal correlations among microscopical elements of a lower scale level;
(4) during nonmonotonous acceleration of a plate when an acceleration changes sign at $t = t^*$ and $\dot{U}(t^*) = 0$, the shift model parameter γ goes to zero at a nonclassical value of the friction on a plate $\Delta(t) \neq 1$. Formally the non-slip boundary condition on a plate ceases to be fulfilled, and there arises a slip of the viscous medium on a plate. From the mathematical point of view this situation may be classified with the degenerated one when $\varepsilon \to 0$. It means that the obtained

approximate solution can essentially differ from the rigorous solution, the approximate solution may not exist in general, or a new solution may appear. However, the degeneration does not emerge if a slip velocity on a plate exists as a model parameter. Then the shear solution will be considered to be the first-order approximate solution. In both cases the nonclassical solutions can exist only if some conditions on the given parameters $\dot{U}$, Δ, ε are fulfilled simultaneously. So, it must be emphasized that after a critical moment t^* the two types of nonclassical solutions can exist: vortical formations or rotations ($\gamma \neq 0$), and shear with a slip ($\gamma = 0$).

The results enumerated above allow explanation of the experiments described in Section 2. When a shock wave front propagates through a material, stress fields with large space gradients and deformation velocities arise. Herewith material structure elements are caused to move in the direction of the wave front propagation. Due to the structure inhomogeneity each element moves as individual microflow characterized by their velocity, density, temperature and viscosity. Such movement of a material corresponds to the mesostructure level described in Section 2. At high velocities a viscous interaction between microflows plays an important role. In order to deal with a viscous shear flow near the boundary of the adjacent microflows, neglecting their collective interaction and the edge effects, and considering this boundary to be a plane moving in a viscous medium, we come to the model state of the Raleigh problem. For the sake of simplicity assume a limiting case where microflows differ from each other by the values of densities and viscosities and consider one of them to be solid and the other to be liquid. The inhomogeneous behavior of the dispersion in microflow velocities during the wave front propagation has been experimentally found in Section 2. It means that the plate acceleration relative to the liquid also has to be inhomogeneous. For the one-dimensional problem in a whole space without edge effects the nonlocality parameter ε corresponding to the relative microflow size should be considered small but finite $\varepsilon \ll 1$. Under conditions the results enumerated in the four conclusions above and corresponding to the slightly nonlocal approximation (see Section 3.4 and 4.2) are valid.

So, at finite accelerations the nonlocal effects cause the medium polarization and rotations of structure elements with increase in accelerations $\dot{U}$. When the critical moment $t = t^*$ is reached one of the three probable solutions exists depending on combinations of the parameters U, Δ, ε:

(1) in the classical solution to the Raleigh problem for continuum no structure formations are seen;
(2) the vortical structures or rotations described above emerge;
(3) a shear of a plate with a slip relative to a viscous liquid arises.

It must be noticed that this shear arises only after the critical moment $t = t^*$ when rotations had formed. In alternative case new structure formations do not occur. The space distributions of values $\dot{U}$, Δ, ε over a material has stochastic character, hence the conditions under which all nonclassical solutions emerge, are also probable. This fact has been experimentally confirmed by the results described in Section 2.

6. Summary

Within the scope of the developed self-consistent nonlocal hydrodynamical theory of non-equilibrium transport processes first we succeeded in determining direct relationships between the nonequilibrium effects of memory and nonlocality leading to internal structure of medium. Accelerations were found to cause the emergence of nonclassical solutions to the problem on a viscous shear. These solutions correspond to the probable formation of the new space structures in materials during high-strain-rate processes.

References

Aero, E.L., 1981. Fluid mechanics. Soviet Research 10 (4).

Alekseevskii, V.P., 1966. Penetration of a rod into a target at high velocity. Fiz. Goren. Vzryva 2, 99.

Balankin, A.S., 1991. Synergetics of a Deformed Solid. Part 1. Ministry of Defence, USSR. Moscow.

Barakhtin, B.K., Mescheryakov, Yu. I., Savenkov, G.G., 1991. Microstructure of materials and model of high-velocity penetration of plane cumulative streams. J. Tech. Phiz. 61 (6), 8–11.

Bellout, H., Bloom, F., Neĉas, J., 1992. Phenomenological behaviour of multipolar viscous fluids. Quart. Appl. Math. 50, 559–883.

Filippov, B.V., Khatuleva, T.A., 1984. Boundary problems of nonlocal hydrodynamics. Leningrad State University, in Russian.

Fursenko, A.A., Khantuleva, T.A., Makarov, Yu. N., Ramm, M.S., 1992. Mathematical simulation of transport processes of species in gas phase in modern sputter-deposition technologies of thin films. 1. Formulation of the problem. Preprint 1595, Ioffe Phys. Tech. Inst., St Petersburg.

Green, A.C., Rivlin, R.S., 1964. Multipolar continuum mechanics. Archives for Rational Mechanics and Analysis 17, 113–147.

Hohler, V., Stilp, A.J., 1990. Long-rod penetration mechanics. In: Zukas J.A. (Ed.), High Velocity Impact Dynamics, ISBNO-471-51444-G. John Wiley and Sons.

Khantuleva, T.A., 1982. Nonlocal hydrodynamical models of gas flows in the transition regime. In Papers from the 13th Int. Symp. on RGD 1, 229–236. Plenum Press.

Khantuleva, T.A., Vavilov, S.A., 1994. Mathematical analysis of nonlocal hydrodynamical model for rarefied gases. In Book of Abstr. of the 19th Int. Symp. on RGD, Oxford University, U.K.

Likhachev, V.A., Panin, V.E., Zosimchuk, E.E. et al., 1989. Cooperative Deformation Processes and Localization. Naukova Dumka, Kiev, p. 320.

Mescheryakov, Yu. I., Atroshenko, S.A., Vasilkov, V.B., Chernyshenko, A.I., 1992. Criteria of transition from translational to rotational motion of media under shock loading. In: Schmidt, S.C., Dick, R.D., Forbes, J.W., Tasker, D.C. (Eds.), Shock Compression of Condensed Matter, pp. 407–410. Elsevier.

Mescheryakov, Yu. I., Atroshenko, S.A., 1992. Multiscale rotations in dynamically deformed solids, I. International Journal of Solids and Structures 29 (22), 2761–2778.

Mescheryakov, Yu. I., Makhutov, N.A., Atroshenko, S.A., 1994. Micromechanisms of dynamic fracture of ductile high-strength steel. Journal of the Mechanics and Physics of Solids 42 (9), 1435–1557.

Piccirelli, R., 1968. Theory of the dynamics of simple fluid for large spatial gradient and long memory. Physics Review 175 (1), 77–98.

Richardson, J.M., 1960. The hydrodynamical equations of a one component system derived from nonequilibrium statistical mechanics. J. Math. Anal., Appl. 1, 12–60.

Savenkov, G.G., Mescheryakov, Yu. I., Vasilkov, V.B., Chernyshenko, A. I., 1990. Grain oscillations and development of turbulent stage of plastic deformation under high-velocity interaction of solids. Fiz. goren. i. vzryva 5, 97–102.

Tate, A., 1967. A theory for the deceleration of long rods after impact. Journal of Mechanics and Physics of Solids 15, 387.

Vavilov, S.A., 1990. On the solvability of one class of boundary value problems. Differential and Integral Equations 3 (1), 175–179.

Vavilov, S.A., 1992. Geometric methods of studying the solvability of a class of operator equations. Russian Acad. Sci. Dokl. Math. 45 (2), 276–280.

Vavilov, S.A., Yuhnevich, S.V., 1993. On the solvability of a class of boundary value problems. Nonlinear Vibration Problems 25, 475–482.

Zlatin, N.A., Koshushko, A.A., 1982. Hydrodynamical limit of cumulation. J. Tech. Phys. 2.

Zubarev, D.N., Tischenko, S.V., 1972. Nonlocal hydrodynamics with memory. Physica 59 (2), 285–304.